Problem 6
Prove or give a counterexample.
(a) Every oscillating sequence has a convergent subsequence.
(b) Every oscillating sequence diverges.
(c) Every divergent sequence oscillates.

Solution
According to the text (page 184), an oscillating sequence \( (s_n) \) is a bounded sequence for which \( \lim \inf s_n < \lim \sup s_n \).

(a) With the above definition in mind we see that part a is trivially true, as any bounded sequence has a convergent subsequence.

(b) We note that if \( (s_n) \) converges (say, to \( s \)), then all its subsequences converge (also, to \( s \)), so \( \lim \inf(s_n) = \lim \sup(s_n) \). Therefore, if \( (s_n) \) oscillates, we know \( \lim \inf(s_n) < \lim \sup(s_n) \), and it follows that \( (s_n) \) does not converge. That is, \( (s_n) \) diverges. Therefore, part b is true as well.

(c) The sequence \( (s_n) \) defined by \( s_n = n \) diverges to infinity, but does not oscillate, as a requirement for an oscillating series is that it be bounded. So if the term divergent, would allow for diverges to infinity, then, as stated, part c is false. But what if the intent is to disallow series that diverge to \( \pm \infty \)? In that case, part (c) is true, as we claim that a divergent (bounded) series has \( \lim \inf s_n < \lim \sup s_n \).

Proof of claim:
Suppose that \( (s_n) \) is a divergent bounded series. The the contrapositive of the above claim is that, if \( \lim \inf s_n \geq \lim \sup s_n \), then \( (s_n) \) converges. As it is impossible that \( \lim \inf s_n > \lim \sup s_n \), this reduces to showing that, if \( \lim \inf s_n = \lim \sup s_n \), then \( (s_n) \) converges. We note that this statement is the same as exercise 19.9 on page 189. To simplify notation, let us say that \( \lim \inf s_n = \lim \sup s_n = s \). We claim that then \( (s_n) \) converges to \( s \). Let \( \varepsilon > 0 \) be given. By Theorem 19.11 \( \exists N_1 \ni n > N_1 \Rightarrow s_n < s + \varepsilon \). By the extension to this theorem discussed in class (regarding \( \lim \inf \)), \( \exists N_2 \ni n > N_2 \Rightarrow s - \varepsilon < s_n \). Choose \( N = \max\{N_1, N_2\} \). If \( n > N \) then \( s - \varepsilon < s_n < s + \varepsilon \), or \( |s_n - s| < \varepsilon \).

終わりました