Page 208 Problem 17

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}. \qquad (*)$$

Prove that there exists $k \in \mathbb{R}$ such that $f(x) = kx$, for every $x \in \mathbb{R}$.

Solution

Let $k = f(1)$. We will show that $f(x) = kx$ for all $x \in \mathbb{R}$ by breaking the solution into parts in accordance with the hints given for this exercise on page 373 (and, indirectly, on page 372).

1. We claim that, for $x \in \mathbb{R}$, and $n \in \mathbb{N}$, $f(nx) = nf(x)$.
The claim is trivially true when $n = 1$. Suppose, for $n \geq 1$, that $f(nx) = nf(x)$. Then

$$\begin{aligned} f((n+1)x) &= f(nx + x) \\ &= f(nx) + f(x) && \text{(by *)} \\ &= nf(x) + f(x) && \text{(by the induction assumption)} \\ &= (n+1)f(x). \end{aligned}$$

2. We now show that, for all $n \in \mathbb{N}$ and $y \in \mathbb{R}$, $f\left(\frac{y}{n}\right) = \frac{f(y)}{n}$.
This follows by letting $\frac{y}{n}$ take the role of x in (1).

3. Next, we show that, for all $r \in \mathbb{Q}^+$, $f(r) = f(1)r = kr$.
Let $r \in \mathbb{Q}^+$. Then there exists $m, n \in \mathbb{N}$ such that $r = \frac{m}{n}$. Thus,

$$\begin{aligned} f(r) &= f\left(\frac{m}{n}\right) = f\left(m\frac{1}{n}\right) \\ &= mf\left(\frac{1}{n}\right) && \text{(by 1, with } m \text{ taking the role of } n\text{)} \\ &= \frac{m}{n}f(1) && \text{(by 2, with 1 taking the role of } y\text{)} \\ &= rk. && \text{(by definition of } k\text{)} \end{aligned}$$

4. Next we observe that, for $x \in \mathbb{R}^+$, $f(x) = kx$.
Let $x \in \mathbb{R}^+$, and let (r_n) be a sequence of positive rationals that converges to x. Then

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f(r_n) && \text{(because } f \text{ is continuous)} \\ &= \lim_{n\to\infty} kr_n && \text{(by 3)} \\ &= k \lim_{n\to\infty} r_n && \text{(basic limit property)} \\ &= kx. \end{aligned}$$

5. If $x = 0$, then we also claim $f(x) = kx$, i.e., we claim $f(0) = 0$.
This follows quickly from (*) because $f(0) = f(0+0) = f(0) + f(0)$.

6. Finally, we show that, for $x \in \mathbb{R}^-$, $f(x) = kx$.
Clearly, $f(0) = 0 = f(x - x) = f(x + (-x)) = f(x) + f(-x)$. Thus, for $x \in \mathbb{R}^-$,

$$\begin{aligned} f(x) &= -f(-x) && \text{(by what we have just said)} \\ &= -k(-x) && \text{(by 4, because } -x \in \mathbb{R}^+\text{)} \\ &= kx. \end{aligned}$$

Voilà!