The Work of Vito Volterra (1860–1940)
See *The Calculus Gallery*, by William Dunham, pp. 170–182

Photos courtesy of [http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Volterra.html](http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Volterra.html)

Vito Volterra: Early Career
Two Great Results (1881, at age 21!)

Vito Volterra: Mid Career

Vito Volterra: Later Years
Vito Volterra: Early Career

- Born in Ancona, Italy
Vito Volterra: Early Career

- Born in Ancona, Italy
- Raised in Florence
Vito Volterra: Early Career

- Born in Ancona, Italy
- Raised in Florence
- A true “Renaissance Man”
Vito Volterra: Early Career

- Ph.D. (Physics) at age 22

\[
\begin{align*}
\frac{dx}{dt} &= x(a - by) \\
\frac{dy}{dt} &= -y(c - dx)
\end{align*}
\]

- $x$: number of prey
- $y$: number of predators
- $t$: time
- $a$, $b$, $c$, and $d$ are constants
Vito Volterra: Early Career

- Ph.D. (Physics) at age 22
- Published in Biology
  (predator-prey equations):
  \[
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  \]
  \[x = \text{number of prey}\]
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Vito Volterra: Early Career
Two Great Results (1881, at age 21!)
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Two Great Results (1881, at age 21!)

1. Constructed a function $f$ whose derivative $f'$ exists everywhere, but the (Riemann) integral $\int_a^b f'(x) \, dx$ does not exist.

Thus (for the Riemann integral), if one is to use the formula $\int_a^b f'(x) \, dx = f(b) - f(a)$, one must first be convinced that the (Riemann) integral of the derivative exists.

2. Proved that there cannot exist two pointwise discontinuous functions on the interval $(a, b)$ for which the continuity points for one are the discontinuity points for the other, and vice versa.

Application: It is impossible for a function to be continuous on the rationals and discontinuous on the irrationals because the "ruler function" defined by $R(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$ is continuous on the irrationals and discontinuous on the rationals.
Two Great Results (1881, at age 21!)

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Corollary: There does not exist a continuous function \( g \) defined on the real numbers such that \( g(x) \) is irrational when \( x \) is rational and \( g(x) \) is rational when \( x \) is irrational.

Proof:
Suppose such a function \( g \) existed and consider \( G(x) = R(g(x)) \), where \( R \) is the ruler function. We claim (proof below) that the function \( G \) so defined would be continuous on the rationals and discontinuous on the irrationals. But Volterra proved that there can be no such function, so the supposition that the function \( g \) exists is incorrect.

Proof of Claim:
**Corollary:** There does not exist a continuous function $g$ defined on the real numbers such that $g(x)$ is irrational when $x$ is rational and $g(x)$ is rational when $x$ is irrational.

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**Proof of Claim:**
- Suppose that $x_0$ is rational. Then by supposition $g(x_0)$ is irrational. The ruler function is continuous on the irrationals and $g$ is continuous everywhere, so the composition $G(x) = R(g(x))$ is continuous at $x = x_0$. 


**Corollary:** There does not exist a continuous function $g$ defined on the real numbers such that $g(x)$ is irrational when $x$ is rational and $g(x)$ is rational when $x$ is irrational.

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**Proof of Claim:**

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- **Suppose that** $x_0$ **is irrational and let** $(x_n)$ **be a sequence of rational numbers that converges to** $x_0$. **Then, for all** $n$, **$g(x_n)$ is irrational, so** $R(g(x_n)) = 0$. **Thus,**
  \[
  \lim_{x_n \to x_0} G(x_n) = \lim_{x_n \to x_0} R(g(x_n)) = \lim_{x_n \to x_0} 0 = 0.
  \]

  However, $x_0$ is irrational, so $g(x_0)$ is some rational number, say $g(x_0) = \frac{p}{q}$ in lowest terms. Then $G(x_0) = R(g(x_0)) = R\left(\frac{p}{q}\right) = \frac{1}{q} \neq 0$.

  Therefore, $G$ is discontinuous at $x_0$ because $0 = \lim_{x_n \to x_0} G(x_n) \neq G(x_0) = \frac{1}{q}$. 

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- Volterra publicly opposed Mussolini in the 1920s. In 1931 he refused to take a mandatory oath of loyalty to Mussolini (only 12 out of 1,250 professors refused).
- This stance cost him his job!
- He lived abroad for most of the rest of his life.
According to Wikipedia Volterra was not a political radical, and would likely have opposed a leftist regime as well. Wikipedia cites a quotation of Volterra’s found on a postcard as an apt description of his political philosophy: “Empires die, but Euclid’s theorems keep their youth forever.”
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Volterra eventually received an honorary knighthood by Britain’s King George V.
Vito Volterra: Later Years

- Volterra eventually received an honorary knighthood by Britain’s King George V.
- Volterra reflected on the 1800s as “the century of the theory of functions.”